Int. J. Heat Mass Transfer. Vol. 18, pp. 237–244. Pergamon Press 1975. Printed in Great Britain

THE EFFECTS OF BLOWING AND SUCTION ON FREE CONVECTION BOUNDARY LAYERS

J. H. MERKIN
Lecturer, School of Mathematics, University of Leeds, Leeds LS2 9JT, England

(*Received* 22 *April* 1974 *and in revised form* 31 *May* 1974)

Abstract—The effects of blowing and suction on free convection boundary layers on bodies of general shape are considered. A numerical solution is obtained for a horizontal circular cylinder with both constant blowing and suction. The particular body shapes and transpiration velocities which enable a similarity solution of the boundary-layer equations to be found are given. The asymptotic solutions both strong blowing and suction are derived and it is found that there is good agreement with the numerical solutions.

NOMENCLATURE

X, co-ordinate measuring distance along body;
Y, co-ordinate measuring distance normal to body;
V_0, transpiration velocity;
l, typical length scale of the body;
g, acceleration of gravity;
T, temperature of fluid;
T_0, ambient temperature;
T_1, temperature of body;
ΔT, $= T_1 - T_0$;
α, angle made by outward normal with downward vertical;
$S(x)$, $= \sin\alpha$;
$\gamma(x)$, non-dimensional variation in transpiration velocity;
ν, kinematic viscosity;
Ψ, stream function;
ε, transpiration parameter;
σ, Prandtl number;
Q, Q^*, heat-transfer parameters;
$\tau_\omega, \tau_\omega^*$, skin friction parameters;
Gr, Grashof number;
β, coefficient of thermal expansion.

1. INTRODUCTION

PREVIOUS work on the effects of suction and blowing on free convection boundary layers has been confined almost entirely to the case of a heated vertical plate. Eichhorn [1] considered the power law variations in plate temperature and transpiration velocity which enable a similarity solution of the boundary-layer equations to be found. Sparrow and Cess [2] discussed the case of constant plate temperature and transpiration velocity. They obtained series expansions for temperature and velocity in powers of $x^{1/4}$, x being the co-ordinate measuring distance from the leading edge. Merkin [3] extended this problem by obtaining asymptotic expansions, i.e. as $x \to \infty$, for temperature and velocity in the cases both of blowing and suction. The series expansions for small x given in [2] were then joined to the asymptotic solution by a numerical solution of the boundary-layer equations.

In a recent paper, Clarke [4] extended the problem discussed in [1] by obtaining the next approximation to the solution of the full Navier–Stokes equations for large, but finite, Grashof number. In [4] the density variations were included in full, whereas in [1–3] the density variations were assumed important only in producing the buoyancy force.

The only paper not limiting attention to a flat plate is by Aroesty and Cole [5]. They considered the case of strong blowing through a body of general shape, and obtained the first approximation in the inner inviscid region, but did not extend it any further, or discuss the form of the outer region which is needed so that the ambient conditions can be attained by the fluid.

The purpose of this paper is to consider the effects that blowing and suction have on free convection boundary layers on bodies of general shape. A general transpiration velocity is considered and attention is limited to two-dimensional flows. The forms of body shape and transpiration velocity which enable a similarity solution of the equations to be obtained is discussed. A method of solving the full boundary-layer equations numerically is considered and results given for a horizontal circular cylinder with constant transpiration velocity. Solutions are then obtained for large values of a transpiration parameter ε. For the case of suction, series expansions for velocity and temperature are obtained in powers of ε^{-4}, while for blowing the expansions are in powers of ε^{-2}. A comparison of the exact values obtained from the numerical solution with those obtained from the asymptotic series gives good agreement even for moderate values of ε. This suggests that the asymptotic series would be useful to describe the flow in cases other than those for which numerical solutions have been obtained.

2. EQUATIONS OF MOTION

The co-ordinate system used is such that X measures distance along the surface of the body, the lowest point being the origin $X = 0$, and Y measures distance normally outwards. With the assumption that imposed temperature differences are small, viscous dissipation can be neglected and changes in density are important only in producing the buoyancy force. The boundary-layer equations are as given in, for example, [3] except that the transverse component of the buoyancy force is $g\beta(T-T_0)\sin\alpha$, where α is the angle made by the outward normal with the downward vertical. The transpiration velocity on the body is $\pm V_0\gamma(x)$, where the upper sign is taken throughout for blowing and the lower sign for suction, and $\gamma(x)$ is non-dimensional.

The continuity equation enables a stream function Ψ to be used and defining non-dimensional variables $x = X/l$, $y = Gr^{1/4}Y/l$, $\Psi = \nu Gr^{1/4}\psi(x,y)$ and $T-T_0 = \Delta T\theta(x,y)$, where l is a typical length scale of the body and $Gr = g\beta\Delta Tl^3/\nu^2$ is the Grashof number, the boundary-layer equations become

$$\frac{\partial^3\psi}{\partial y^3} + S(x)\theta + \frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2} - \frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial x\partial y} = 0 \qquad (1)$$

$$\frac{1}{\sigma}\frac{\partial^2\theta}{\partial y^2} - \frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x} + \frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y} = 0 \qquad (2)$$

with boundary conditions

$$\begin{aligned} &\theta = 1, \quad \frac{\partial\psi}{\partial y} = 0, \quad \frac{\partial\psi}{\partial x} = \mp\varepsilon\gamma(x) \quad \text{on} \quad y = 0 \\ &\theta \to 0, \quad \frac{\partial\psi}{\partial y} \to 0 \quad \text{as} \quad y \to \infty. \end{aligned} \qquad (3)$$

$\varepsilon = Re/Gr^{1/4}$, where $Re = V_0 l/\nu$ is the Reynolds number of the applied transpiration flow, and σ is the Prandtl number.

If $S(x) = [(2-\beta)x]^{(3\beta-2)/(2-\beta)}$ then it has been shown by [8] that equations (1) and (2) have a similarity solution without transpiration, where β must be in the range $\frac{2}{3} \leqslant \beta < 2$. If transpiration effects are included a similarity solution is still possible provided $\gamma(x) = [(2-\beta)x]^{(\beta-1)/(2-\beta)}$. Defining

$$\psi = \mp\varepsilon\int\gamma(x)\,dx + [(2-\beta)x]^{1/(2-\beta)}F(\eta_1), \quad \theta = \theta(\eta_1)$$

where

$$\eta_1 = y[(2-\beta)x]^{(\beta-1)/(2-\beta)},$$

equations (1) and (2) become

$$F''' + \theta + (F \mp \varepsilon)F'' - \beta F'^2 = 0 \qquad (4)$$

$$\theta'' + \sigma(F \mp \varepsilon)\theta' = 0 \qquad (5)$$

with

$$\begin{aligned} &F(0) = F'(0) = 0, \quad \theta(0) = 1 \\ &F' \to 0, \quad \theta \to 0 \quad \text{as} \quad \eta_1 \to \infty. \end{aligned} \qquad (6)$$

(Dashes denote differentiation with respect to η_1.)

The case $\beta = 1$ gives the only body shape for which a similarity solution is possible with a constant transpiration velocity. In this case $S(x) = x$ and $F(\eta_1)$ and $\theta(\eta_1)$ are the initial profiles for the constant transpiration flow on a general body with a rounded lower end. The semi-infinite flat plate is given by $\beta = \frac{2}{3}$, and the solution of the equations in this case have been given by [1] and [4]. Graphs of $Q^* = -\varepsilon^{-1}\theta'(0)$ and $\tau^*_\omega = \varepsilon F''(0)$ for $\beta = 1$ and $\sigma = 1$ are given in Figs. 1 and 2 for suction and blowing respectively.

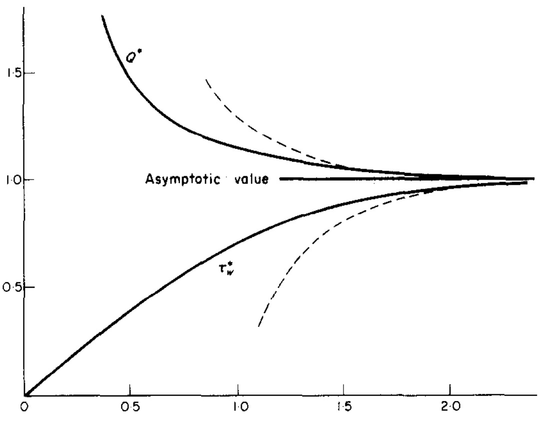

FIG. 1. Heat transfer Q^* and skin friction τ^*_ω for suction, $\beta = 1$. ——— exact values, — — — values from asymptotic expansion.

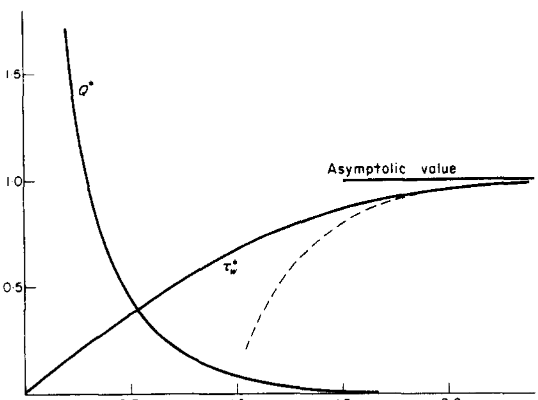

FIG. 2. Heat transfer Q^* and skin friction τ^*_ω for blowing, $\beta = 1$. ——— exact values, — — — values from asymptotic expansion.

Equations (1) and (2) have been solved numerically for a constant transpiration velocity for a horizontal circular cylinder, in which case $S(x) = \sin x$. The method of solution is similar to that described in [3] for a flat plate. In [3] the equations were first transformed using a transformation appropriate for a body with a sharp leading edge, and then these equations were solved numerically. This transformation is not suitable for a body with a rounded lower end, for which $S(x)/x \to 1$ as $x \to 0$. The transformation appropriate in this case is

$$\psi = \mp\varepsilon \int_0^x \gamma(t)\,dt + x\psi_1(x, y),$$

and then the resulting equations are solved for ψ_1 and θ. To do this, derivatives in the x-direction are replaced by differences and all other quantities averaged. The two non-linear ordinary differential equations which result are solved by writing them in finite difference form and solving the algebraic equations iteratively by a Newtonian–Raphson process. The numerical solution starts at $x = 0$ where the initial profiles are given by equations (4) and (5) with $\beta = 1$ and proceeds round the cylinder to $x = \pi$. Errors from using finite differences in the x-direction were kept small by covering each step in first one and then two integrations and insisting that the difference between the two solutions was less than 0·0005. Errors from using finite differences in the y-direction were reduced by doing each calculation twice, first with a step length of h and then with a step length of $2h$. Since the finite difference scheme gives errors of $O(h^2)$, the Richardson h^2 extrapolation was used to improve the results. This method is fully described in [6]. For suction $h = 0{\cdot}1$ and for blowing $h = 0{\cdot}2$ were used. The outer boundary conditions were taken at a finite value of $y = y_\infty$, and y_∞ had to be varied from 7 to 10 for suction, and from 20 to 28 for blowing.

We can define a skin friction parameter τ_ω and a heat-transfer parameter Q by

$$\tau_\omega = \frac{V_0}{g\beta\Delta T}\left(\frac{\partial^2\Psi}{\partial Y^2}\right)_0 = \varepsilon\left(\frac{\partial^2\psi}{\partial y^2}\right)_0$$

and

$$Q = -\frac{\nu}{V_0\Delta T}\left(\frac{\partial T}{\partial Y}\right)_0 = -\frac{1}{\varepsilon}\left(\frac{\partial\theta}{\partial y}\right)_0.$$

Values of τ_ω and Q for suction and blowing for various values of ε and $\sigma = 1$ are given in Tables 1 and 2.

Table 1. Values of skin friction parameter τ_ω and heat-transfer parameter Q for suction

	$\varepsilon = 0{\cdot}5$		$\varepsilon = 1$		$\varepsilon = 2$		$\varepsilon = 2$ (series)	
x	τ_ω	Q	τ_ω	Q	τ_ω	Q	τ_ω	Q
0	0	1·4606	0	1·1160	0	1·0169	0	1·0136
0·5	0·1928	1·4502	0·3416	1·1107	0·4567	1·0158	0·4571	1·0126
1·0	0·3542	1·4184	0·6233	1·0957	0·8094	1·0119	0·8114	1·0093
1·5	0·4566	1·3640	0·7934	1·0700	0·9863	1·0060	0·9850	1·0031
2·0	0·4801	1·2838	0·8146	1·0320	0·9219	0·9985	0·9288	0·9948
2·5	0·4122	1·1689	0·6658	0·9764	0·6393	0·9872	0·6318	0·9869
3·0	0·2418	0·9838	0·3280	0·8758	0·1526	0·9852	0·1517	0·9826
π	0·1619	0·8955	0·1810	0·8100	0·0000	0·9849	0·0000	0·9823

Table 2. Values of skin friction parameter τ_ω and heat-transfer parameter Q for blowing

	$\varepsilon = 0{\cdot}5$		$\varepsilon = 1$		$\varepsilon = 2$		$\varepsilon = 2$ (series)
x	τ_ω	Q	τ_ω	Q	τ_ω	Q	τ_ω
0	0	0·4602	0	0·0769	0	0·0018	0
0·5	0·1880	0·4004	0·3314	0·0749	0·4547	0·0017	0·4531
1·0	0·3435	0·3799	0·5999	0·0691	0·8078	0·0014	0·8131
1·5	0·4379	0·3458	0·7517	0·0598	0·9766	0·0010	0·9931
2·0	0·4509	0·2997	0·7496	0·0471	0·9145	0·0006	0·9329
2·5	0·3707	0·2340	0·5765	0·0318	0·6223	0·0002	0·6284
3·0	0·1875	0·1461	0·2443	0·0146	0·1526	0·0001	0·1499
π	0·1380	0·1253	0·0882	0·0075	0·0000	0·0000	0·0000

3. ASYMPTOTIC EXPANSION—SUCTION

To obtain a solution of equations (1) and (2) which will hold for large ε, new variables are defined by

$$\psi = \varepsilon \int_0^x \gamma(t)\,dt + S\gamma^3\varepsilon^3 f(x,\eta), \quad \theta = \theta(x,\eta)$$

where $\eta = \varepsilon\gamma y$. On substituting into equations (1) and (2) an expansion for f and θ of the form

$$\begin{aligned} f(x,\eta) &= f_0(\eta) \\ &\quad + \varepsilon^{-4}\left(\frac{S}{\gamma^5}\frac{d\gamma}{dx}f_{10}(\eta) + \frac{1}{\gamma^4}\frac{dS}{dx}f_{11}(\eta)\right) + \ldots \\ \theta(x,\eta) &= \theta_0(\eta) \\ &\quad + \varepsilon^{-4}\left(\frac{S}{\gamma^5}\frac{d\gamma}{dx}\theta_{10}(\eta) + \frac{1}{\gamma^4}\frac{dS}{dx}\theta_{11}(\eta)\right) + \ldots \end{aligned} \tag{7}$$

is suggested. On solving the resulting ordinary differential equations

$$\theta_0(\eta) = e^{-\sigma\eta}$$

$$f_0(\eta) = \frac{(\sigma-1) + e^{-\sigma\eta} - \sigma e^{-\eta}}{\sigma^2(\sigma-1)}, \quad (\sigma \neq 1)$$

$$\text{or} \quad 1 - e^{-\eta} - \eta e^{-\eta}, \quad (\sigma = 1)$$

$$\begin{aligned} \theta_{10}(\eta) &= \frac{3}{\sigma}e^{-\sigma\eta}\left(\eta - \frac{2\sigma^2+2\sigma+1}{2\sigma(\sigma+1)}\right) \\ &\quad + \frac{3e^{-\sigma\eta}}{(\sigma-1)}\left(\frac{\sigma}{\sigma+1}e^{-\eta} - \frac{e^{-\sigma\eta}}{2\sigma^2}\right) \quad (\sigma \neq 1) \\ &= 3e^{-\eta}(\eta - \tfrac{5}{4}) + \tfrac{3}{2}e^{-2\eta}(\tfrac{5}{2}+\eta) \quad (\sigma = 1) \end{aligned}$$

$$\theta_{11}(\eta) = -\tfrac{1}{3}\theta_{10}(\eta)$$

$$\begin{aligned} f_{10}(\eta) &= -\frac{3\eta}{\sigma^3(\sigma-1)}(e^{-\eta} - e^{-\sigma\eta}) + \frac{3(5+4\sigma-2\sigma^2)}{2\sigma^4(\sigma+1)(\sigma-1)} \\ &\quad \times (e^{-\sigma\eta} + \sigma - 1) + \frac{(3\sigma^5 - 3\sigma^4 + 3\sigma^3 - \sigma^2 - \sigma + 3)}{\sigma^4(\sigma-1)^2(\sigma+1)^3} \\ &\quad \times (e^{-(1+\sigma)\eta} + \sigma) - \frac{(e^{-2\eta}+1)}{4\sigma^2(\sigma-1)^2} \\ &\quad - \frac{(3\sigma-1)}{8\sigma^4(2\sigma-1)(\sigma-1)^2}(e^{-2\sigma\eta} + 2\sigma - 1) \\ &\quad - \frac{(20\sigma^4 + 107\sigma^3 - 8\sigma^2 - 47\sigma + 12)}{4\sigma^4(\sigma-1)(\sigma+1)^2(2\sigma-1)}e^{-\eta} \quad (\sigma \neq 1) \\ &= \frac{47}{16} - \left(4 + \frac{21}{4}\eta + 3\eta^2\right)e^{-\eta} \\ &\quad + \left(\frac{17}{16} - \frac{5}{8}\eta + \frac{\eta^2}{4}\right)e^{-2\eta} \quad (\sigma = 1) \end{aligned}$$

$$\begin{aligned} f_{11}(\eta) &= \frac{(2\sigma^2 - 4\sigma - 5)}{2\sigma^4(\sigma+1)(\sigma-1)}(e^{-\sigma\eta} - \sigma e^{-\eta} + \sigma - 1) \\ &\quad + \frac{1}{8\sigma^4(\sigma-1)(2\sigma-1)}(e^{-2\sigma\eta} - 2\sigma e^{-\eta} + 2\sigma - 1) \\ &\quad - \frac{(\sigma^4 + \sigma^2 - 1)}{\sigma^4(\sigma+1)^3(\sigma-1)}(e^{-(1+\sigma)\eta} - (1+\sigma)e^{-\eta} + \sigma) \\ &\quad + \frac{\eta(e^{-\eta} - e^{-\sigma\eta})}{\sigma^3(\sigma-1)} \quad (\sigma \neq 1) \\ &= \left(\eta^2 + \frac{7\eta}{4} + \frac{13}{4}\right)e^{-\eta} \\ &\quad - \left(\frac{\eta}{8} + \frac{13}{16}\right)e^{-2\eta} - \frac{39}{16} \quad (\sigma = 1). \end{aligned}$$

From the above solution it follows that asymptotic series for Q and τ_ω are

$$Q = \gamma\left[\sigma + \frac{\varepsilon^{-4}}{2\sigma(\sigma+1)}\left(\frac{1}{\gamma^4}\frac{dS}{dx} - \frac{3S}{\gamma^5}\frac{d\gamma}{dx}\right) + \ldots\right] \tag{8}$$

$$\begin{aligned} \tau_\omega &= \frac{S}{\gamma}\left[\frac{1}{\sigma} + \frac{\varepsilon^{-4}}{4\sigma^3(\sigma+1)^2}\right. \\ &\quad \left. \times \left((25\sigma+19)\frac{S}{\gamma^5}\frac{d\gamma}{dx} - \frac{(9\sigma+7)}{\gamma^4}\frac{dS}{dx}\right) + \ldots\right]. \end{aligned} \tag{9}$$

The solution for the equations of $O(\varepsilon^{-8})$ has been found in a way similar to that above, but this time four sets of equations have to be solved. The process is straightforward but the algebraic manipulation is very involved. The results are very long and will not be given in order to save space.

Using the forms for $S(x)$ and $\gamma(x)$ necessary for a similarity solution, it follows that

$$Q^* = \sigma + \frac{\varepsilon^{-4}}{2\sigma(\sigma+1)} + \ldots \tag{10}$$

$$\tau_\omega^* = \frac{1}{\sigma} - \varepsilon^{-4}\frac{2(\sigma+1)\beta + (7\sigma+5)}{4\sigma^3(\sigma+1)^2} + \ldots. \tag{11}$$

Values of Q and τ_ω obtained from (8) and (9) for a circular cylinder are given in Table 1 for $\varepsilon = 2$ and $\sigma = 1$, and there is good agreement with the exact values found from the numerical solution. This good agreement was also seen in a comparison of velocity and temperature profiles.

Values of Q^* and τ_ω^* obtained from (10) and (11) for $\beta = 1$ and $\sigma = 1$ are plotted on Fig. 1. The series solution differs from the exact value by less than 10 per cent when $\varepsilon = 1$ for Q^* and when $\varepsilon = 1{\cdot}5$ for τ_ω^*.

It seems reasonable to conclude that the series expansions will be useful in giving a good approximation to velocity and temperature profiles for a general body shape and suction velocity for $\varepsilon \geqslant 2$.

4. ASYMPTOTIC EXPANSION—BLOWING

For strong blowing the boundary layer is made up of two regions. There is an inner region of thickness $O(\varepsilon)$ in which viscous effects are negligible, made up of fluid that has been blown out through the body. This region extends from the body up to the "dividing streamline" which is the streamline that emerged from the body at $x = 0$. Since the ambient conditions are not attained by the fluid on the "dividing streamline", there must be an outer region, centred round the "dividing streamline" at the outer edge of which the ambient conditions are attained, and which must merge with the inner solution. This outer region has thickness $O(1)$ and in it viscous effects are important.

To obtain the solution in the inner region, equations (1) and (2) are first transformed by putting $\psi = \varepsilon\phi$ and $\zeta = \varepsilon^{-1}y$. This transformation is suggested by the fact that the inner region has thickness $O(\varepsilon)$ and is inviscid to a first approximation. Then, instead of (ϕ, θ), (u, θ) are used as new dependent variables, where $u \equiv \partial\phi/\partial\zeta$, and instead of (x, ζ), (x, ξ) are used as new independent variables. ξ is defined to be the point on the body where the streamline ϕ = constant emerged. ϕ is related to ξ by $d\phi = -\gamma(\xi)\,d\xi$. $\xi = x$ is the equation of the body and the equation of the "dividing streamline" is $\xi = 0$.

The form of the transformed equations suggests an expansion of the form

$$\begin{aligned} u &= u_0(x,\xi)+\varepsilon^{-2}u_1(x,\xi)+\varepsilon^{-4}u_2(x,\xi)+\dots \\ \theta &= \theta_0(x,\xi)+e^{-2}\theta_1(x,\xi)+\varepsilon^{-4}\theta_2(x,\xi)+\dots . \end{aligned} \tag{12}$$

The equation for θ_0 is $\partial\theta_0/\partial x = 0$, and since $\theta_0 = 1$ on the body $\theta_0(x,\xi) \equiv 1$. Equating terms of $O(\varepsilon^{-2})$ gives $\partial\theta_1/\partial x = 0$, and since $\theta_1 = 0$ on the body $\theta_1(x,\xi) \equiv 0$. Proceeding inductively, it is easy to see that $\theta_n(x,\xi) \equiv 0$ for $n \geqslant 1$. So that in the inner layer we have

$$\theta(x,\xi) \equiv 1. \tag{13}$$

The solution for u_0 is

$$u_0(x,\xi) = 2^{1/2}\left(\int_\xi^x S(t)\,dt\right)^{1/2} \tag{14}$$

which agrees with the result given in [5]. The solution of the equation obtained by equating terms of $O(\varepsilon^{-2})$ is

$$u_1(x,\xi) = -\frac{1}{\gamma(\xi)}\frac{d}{d\xi}\left[\frac{S(\xi)}{\gamma(\xi)}\right]\frac{1}{u_0(x,\xi)}\int_\xi^x u_0(t,\xi)\,dt. \tag{15}$$

From (14) it follows that $u_0 = 2^{1/2}S(x)(x-\xi)^{1/2}+\dots$ near $\xi = x$, so that u_1 is bounded at $\xi = x$, in fact

$$u_1(x,\xi) = -\frac{1}{\gamma}\frac{d}{d\xi}\left(\frac{S}{\gamma}\right)\frac{2}{3}(x-\xi)+\dots \text{there.} \tag{16}$$

The skin friction parameter τ_ω is given by

$$\tau_\omega = -\left(\frac{1}{\gamma(\xi)}u\frac{\partial u}{\partial\xi}\right)_{\xi=x},$$

so that

$$\tau_\omega = -\frac{1}{2\gamma(x)}\left[\frac{\partial u_0^2}{\partial\xi}+\frac{2}{\varepsilon^2}\frac{\partial(u_0u_1)}{\partial\xi} + \frac{1}{\varepsilon^4}\frac{\partial}{\partial\xi}(u_1^2+2u_0u_2)+\dots\right]_{\xi=x}.$$

From (16) it follows that the term of $O(\varepsilon^{-2})$ does not contribute to τ_ω, so to get the next approximation the term of $O(\varepsilon^{-4})$ has to be considered. It follows that

$$u_2(x,\xi) = -\frac{S}{u_0\gamma^3}\frac{d}{d\xi}\left(\frac{S}{\gamma}\right)(x-\xi)+\dots \text{near } \xi = x$$

so that

$$\tau_\omega = \frac{S}{\gamma}\left[1-\frac{\varepsilon^{-4}}{\gamma(x)}\frac{d}{dx}\left(\frac{S}{\gamma}\right)+\dots\right]. \tag{17}$$

Values of τ_ω obtained from (17) for the case of a circular cylinder with constant blowing are given in Table 2 for $\varepsilon = 2$. There is good agreement with the exact values, the difference being less than 3 per cent.

ζ is related to the (x, ξ) variables by

$$\zeta = \int_\xi^x \frac{\gamma(t)}{u(x,t)}\,dt.$$

The inner solution holds in the region between the body and the "dividing streamline" which is given by $\xi = 0$. If $\zeta = \zeta_0(x)$ is the equation of the "dividing streamline" then

$$\zeta_0(x) = \int_0^x \frac{\gamma(t)}{u(x,t)}\,dt.$$

Since the solution in the outer region must merge with the inner region solution near the "dividing streamline" we need to know the behaviour of ϕ near $\xi = 0$. From (12) and using the behaviour of u_0 and u_1 for small ξ

$$\zeta = \zeta_0 - \left[\frac{\gamma(0)}{2^{1/2}I_0^{1/2}}+\frac{A}{\varepsilon^2}\right]\xi - B\xi^2+\dots \tag{18}$$

where

$$I_0(x) = \int_0^x S(t)\,dt, \quad A = \frac{1}{2}\frac{d}{dx}\left(\frac{S}{\gamma}\right)I_0^{-3/2}\int_0^x I_0(t)^{1/2}\,dt$$

and

$$B = \left[\gamma'(0)+\frac{S(0)\gamma(0)}{2I_0}\right]2^{-1/2}I_0^{-1/2}.$$

The highest terms neglected are $O(\xi^3)$ and $O(\varepsilon^{-2}\xi^2)$. (18) generates the first two terms in the outer region. The terms neglected do not appear in these terms. In the special case of constant blowing and a body with a rounded lower end, for which $\gamma = 1$ and $S(0) = 0$, we must use

$$\zeta = \zeta_0-(2^{-1/2}I_0^{-1/2}+A\varepsilon^{-2})\xi-2^{-1/2}I_0^{-3/2}\xi^3/12+\dots . \tag{19}$$

Inverting (18) and (19) gives

$$\phi = -2^{1/2}I_0^{1/2}(\zeta_0-\zeta)\times\left\{1+\left[\frac{2^{-1/2}I_0^{1/2}}{\gamma(0)^2}3\gamma'(0)+\frac{S(0)\gamma(0)}{I_0}\right](\zeta_0-\zeta)+\ldots\right\}+\varepsilon^{-2}\frac{2AI_0}{\gamma(0)}(\zeta_0-\zeta)+\ldots \quad (20)$$

and

$$\phi = -2^{1/2}I_0^{1/2}(\zeta_0-\zeta)\left[1-\frac{(\zeta_0-\zeta)^2}{6}+\ldots\right]+\frac{2AI_0}{\varepsilon}(\zeta_0-\zeta)+\ldots. \quad (21)$$

The outer region is centred round the "dividing streamline" $y=\varepsilon\zeta_0(x)$, and in it viscous effects are important. This suggests putting $z=y-\varepsilon\zeta_0(x)$, then equations (1) and (2) become

$$\frac{\partial^3\psi}{\partial z^3}+\theta S(x)+\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial z^2}-\frac{\partial\psi}{\partial z}\frac{\partial^2\psi}{\partial x\partial z}=0 \quad (22)$$

$$\frac{1}{\sigma}\frac{\partial^2\theta}{\partial z^2}+\frac{\partial\theta}{\partial z}\frac{\partial\psi}{\partial x}-\frac{\partial\theta}{\partial x}\frac{\partial\psi}{\partial z}=0. \quad (23)$$

The outer boundary conditions are

$$\theta\to 0,\quad \frac{\partial\psi}{\partial z}\to 0 \text{ as } z\to\infty.$$

The inner boundary conditions are obtained by insisting that the solution at the outer edge of the inner layer must merge with the solution at the inner edge of the outer layer. In terms of outer variables (20) and (21) become

$$\psi = 2^{1/2}I_0^{1/2}z-\frac{1}{\varepsilon}\left[3\gamma'(0)+\frac{S(0)\gamma(0)}{I_0}\right]\times\frac{I_0}{\gamma(0)^2}z^2+O(\varepsilon^{-2}) \quad (24)$$

$$\psi = 2^{1/2}I_0^{1/2}z-\frac{1}{\varepsilon^2}2^{1/2}\frac{I_0^{1/2}}{6}\times\left[1+6A\frac{(2I_0)^{1/2}}{z^2}+\ldots\right]z^3+O(\varepsilon^{-3}) \quad (25)$$

(24) and (25) suggests an expansion in the form

$$\psi=\psi_0+\varepsilon^{-\alpha}\psi_1+\ldots$$
$$\theta=\theta_0+\varepsilon^{-\alpha}\theta_1+\ldots \quad (26)$$

where $\alpha=1$ in general, but $\alpha=2$ in the special case. The inner boundary conditions for the outer region are applied on $z=-\varepsilon\zeta_0(x)$, but since we are looking for a solution for large ε, the inner boundary conditions can be applied as $z\to-\infty$ provided that the inner solution is approached through exponentially small terms. So the inner boundary conditions are

$$\psi_0\sim 2^{1/2}I_0^{1/2}z,\quad \theta_0\to 1$$
$$\psi_1\sim\frac{I_0}{\gamma(0)^2}\left[3\gamma'(0)+\frac{S(0)\gamma(0)}{I_0}\right]z^2,\quad \theta_1\to 0 \quad (27)$$

or

$$\left\{\psi_1\sim 2^{1/2}\frac{I_0^{1/2}}{6}z^3\left[1+6A\frac{(2I_0)^{1/2}}{z^2}+\ldots\right]\right\}$$

as $z\to-\infty$.

The first order equations are the same as (22) and (23) with ψ and θ replaced by ψ_0 and θ_0. The resulting equations have to be solved numerically. This has been done for the case of a circular cylinder with constant blowing, where $I_0=1-\cos x$. Values of θ_0 and $\partial\psi_0/\partial z$ are given in Table 3 for various x and $\sigma=1$. There is no problem, in theory, in proceeding to obtain the next order solution, but since the first order solution is given numerically the process involves a long computation and has not been carried out.

To check that the outer solution approaches the inner boundary conditions through exponentially small terms, put $\psi_0=2^{1/2}I_0^{1/2}z+g_0$ and $\theta_0=1+h_0$, where g_0 and h_0 are small. Putting these in the equations for ψ_0 and θ_0, neglecting all but the lowest order terms, and solving the resulting linear equations, gives, for $\sigma=1$,

$$h_0(\eta_0)=A_0\int_{-\infty}^{\eta_0}\frac{e^{-t}}{t^{1/2}}\,dt\sim A_0\frac{e^{-\eta_0}}{\eta_0^{1/2}}$$

for large η_0 where

$\eta_0=z^2/\mathrm{d}(x)$,

and

$$\mathrm{d}(x)=2^{3/2}I_0^{-1}\int_0^x I_0^{1/2}(t)\,dt,$$

and

$$\frac{\partial g_0}{\partial z}=I_0^{1/2}2^{-1/2}h_0(\eta_0)+B_0 2^{-1/2}I_0^{-1/2}(I_0 d)^k e^{-\eta_0}U(k+\tfrac{1}{2},\tfrac{1}{2},\eta_0)$$

for values of $k>0$. $U(k+\frac{1}{2},\frac{1}{2},\eta_0)$ is the confluent hypergeometric function not exponentially large at infinity as given in [7].

To discuss the behaviour of the similarity solution for large ε put $F=\varepsilon\phi$ and $\zeta=\varepsilon^{-1}\eta_1$, then using $u=d\phi/d\zeta$ as a new dependent variable and ϕ as a new independent variable, the expansion for $u(\phi)$ is

$$u(\phi)=u_0(\phi)+\varepsilon^2u_1(\phi)+\ldots$$

where

$$u_0=\beta^{-1/2}[1-(1-\phi)^{2\beta}]^{1/2} \quad (28)$$

$$u_1=-\frac{(2\beta-1)(1-\phi)^{2\beta}}{[1-(1-\phi)^{2\beta}]^{1/2}}\int_{1-\phi}^{\phi}\frac{(1-s^{2\beta})^{1/2}\,ds}{s^3} \quad (29)$$

Table 3. Values of θ_0 and $\partial\psi_0/\partial z$ for asymptotic solution for strong blowing in the case of a horizontal circular cylinder

(a) Values of θ_0

z	$x = 0{\cdot}5$	$x = 1{\cdot}0$	$x = 1{\cdot}5$	$x = 2{\cdot}0$	$x = 2{\cdot}5$	$x = 3{\cdot}0$
−5	1·000	1·000	1·000	1·000	1·000	1·000
−3	0·999	0·999	0·998	0·997	0·995	0·990
−2	0·984	0·982	0·979	0·973	0·964	0·951
−1	0·888	0·884	0·877	0·867	0·854	0·838
0	0·637	0·637	0·638	0·639	0·640	0·643
1	0·337	0·344	0·355	0·372	0·394	0·421
2	0·139	0·146	0·159	0·178	0·205	0·242
3	0·049	0·054	0·062	0·075	0·096	0·128
5	0·005	0·006	0·008	0·011	0·018	0·032
8	0·000	0·000	0·000	0·001	0·001	0·004
11	0·000	0·000	0·000	0·000	0·000	0·000

(b) Values of $\partial\psi_0/\partial z$

z	$x = 0{\cdot}5$	$x = 1{\cdot}0$	$x = 1{\cdot}5$	$x = 2{\cdot}0$	$x = 2{\cdot}5$	$x = 3{\cdot}0$
−5	0·495	0·959	1·363	1·683	1·898	1·995
−3	0·494	0·958	1·362	1·680	1·893	1·984
−2	0·490	0·949	1·347	1·657	1·858	1·936
−1	0·462	0·893	1·263	1·547	1·726	1·788
0	0·377	0·729	1·036	1·275	1·433	1·497
1	0·248	0·486	0·703	0·888	1·027	1·110
2	0·134	0·268	0·400	0·527	0·642	0·735
3	0·062	0·128	0·199	0·277	0·361	0·445
5	0·010	0·023	0·039	0·062	0·094	0·138
8	0·001	0·001	0·002	0·005	0·010	0·019
11	0·000	0·000	0·000	0·000	0·001	0·002

(17) gives, on using the forms for $S(x)$ and $\gamma(x)$ necessary for a similarity solution,

$$\tau_\omega^* = 1-\varepsilon^{-4}(2\beta-1)+\ldots. \quad (30)$$

Values of τ_ω^* for the case $\beta = 1$ are given in Fig. 2. There is good agreement with the exact values, the difference being less than 10 per cent at $\varepsilon = 1{\cdot}4$.

On the "dividing streamline" $\phi = 1$, and if this is given by $\zeta = \zeta_0$ then

$$\zeta_0-\zeta = \beta^{1/2}\left[(1-\phi)+\frac{(1-\phi)^{2\beta+1}}{2(2\beta+1)}\right]\varepsilon^{-2} \times \frac{\beta}{2}(1-\phi)^{2\beta-1}+\ldots. \quad (31)$$

Inverting (31) gives

$$\phi = 1-\beta^{-1/2}(\zeta_0-\zeta)+\frac{\beta^{-(2\beta+1)/2}}{2(2\beta+1)}(\zeta_0-\zeta)^{2\beta+1} + \varepsilon^{-2}\frac{\beta^{1-\beta}}{2}(\zeta_0-\zeta)^{2\beta-1}+\ldots. \quad (32)$$

To get a solution in the outer region put $z = \eta_1-\varepsilon\zeta_0$ then the equations in the outer region are the same as (4) and (5) except that differentiation is now with respect to z. In terms of outer region variables (32) gives

$$F = \varepsilon+\beta^{-1/2}z+\varepsilon^{-2\beta}|z|^{2\beta+1}\frac{\beta^{-(2\beta+1)/2}}{2(2\beta+1)} \times\left[1+\frac{\beta^{3/2(2\beta+1)}}{z^2}+\ldots\right] \quad (33)$$

which suggests an expansion in the form

$$\begin{aligned} F &= \varepsilon+F_0(z)+\varepsilon^{-2\beta}F_1(z)+\ldots \\ \theta &= \theta_0(z)+\varepsilon^{-2\beta}\theta_1(z)+\ldots. \end{aligned} \quad (34)$$

The inner boundary conditions are

$$\theta_0 \to 1 \quad F_0 \sim \beta^{-1/2}z$$

$$\theta_1 \to 0 \quad F_1 \sim \frac{\beta^{-(2\beta+1)/2}}{2(2\beta+1)}|z|^{2\beta+1} \times\left[1+\beta^{3/2}\left(\frac{2\beta+1}{z^2}\right)+\ldots\right]$$

as $z \to -\infty$.

Solution of the first order equations for $\beta = \frac{2}{3}$ have already been given in [3]. The values of F_0', θ_0, F_1' and θ_1 for $\beta = 1$ and $\sigma = 1$ are given in Table 4. Again it can be shown, by a method similar to that given above, that each term approaches the inner conditions through exponentially small terms.

Table 4. Values of F_0', θ_0, F_1' and θ_1 for $\beta = 1$

z	F_0'	θ_0	F_1'	θ_1
−5·0	1·0000	1·0000	−13·0000	0·0000
−0·30	0·9995	0·9991	−5·0050	−0·0070
−2·0	0·9913	0·9841	−2·5340	−0·0409
−1·5	0·9735	0·9531	−1·6806	−0·0641
−1·0	0·9343	0·8882	−1·0569	−0·0729
−0·5	0·8642	0·7804	−0·6228	−0·0561
0·0	0·7614	0·6366	−0·3333	−0·0202
0·5	0·6347	0·4795	−0·1491	0·0156
1·0	0·5006	0·3351	−0·0404	0·0371
1·5	0·3751	0·2197	0·0165	0·0424
2·0	0·2689	0·1370	0·0401	0·0374
3·0	0·1246	0·0482	0·0402	0·0200
5·0	0·0206	0·0050	0·0122	0·0032
7·0	0·0028	0·0005	0·0023	0·0004
9·0	0·0004	0·0001	0·0004	0·0001
11·0	0·0000	0·0000	0·0000	0·0000

REFERENCES

1. R. Eichhorn, The effect of mass transfer on free convection, *J. Heat Transfer* **82**, 260 (1960).
2. E. M. Sparrow and R. D. Cess, Free convection with blowing or suction, *J. Heat Transfer* **83**, 387 (1961).
3. J. H. Merkin, Free convection with blowing and suction, *Int. J. Heat Mass Transfer* **15**, 989 (1972).
4. J. F. Clarke, Transpiration and natural convection: the vertical flat plate problem, *J. Fluid Mech.* **57**, 45 (1973).
5. J. Aroesty and J. D. Cole, Boundary layer flow with large injection rates, Rand Corp. Memorandum RM-4620-ARPA (1965).
6. D. R. Hartree, *Numerical Analysis*. Oxford University Press, Oxford (1952).
7. L. J. Slater, Confluent hypergeometric functions. Cambridge University Press, Cambridge (1960).
8. W. H. Braun, S. Ostrach and J. E. Heighway, Free convection similarity flows about two-dimensional and axisymmetric bodies with closed lower ends, *Int. J. Heat Mass Transfer* **2**, 121 (1961).

LES EFFETS DU SOUFFLAGE ET DE L'ASPIRATION SUR LES COUCHES LIMITES DE CONVECTION NATURELLE

Résumé—On considère les effets du soufflage et de l'aspiration sur la couche limite de convection naturelle autour de corps de forme quelconque. Une solution numérique est obtenue pour un cylindre circulaire horizontal avec un soufflage ou une aspiration constante. On donne la solution de similitude des équations de la couche limite pour des formes particulières d'obstacle et des vitesses de transpiration compatibles. On obtient les solutions asymptotiques pour un soufflage fort ou une aspiration importante et on trouve un bon accord avec les solutions numériques.

GRENZSCHICHTBEEINFLUSSUNG BEI FREIER KONVEKTION DURCH AUSBLASEN UND ABSAUGEN

Zusammenfassung—Es werden die Einflüsse von Ausblasen und Absaugen auf die Grenzschichten von Körpern allgemeiner Form bei freier Konvektion behandelt. Für den horizontalen Kreiszylinder wird eine numerische Lösung für konstantes Ausblasen und Absaugen erhalten. Die besondern Körperformen und Blasgeschwindigkeiten, welche das Auffinden einer Ähnlichkeitslösung ermöglichen, werden angegeben. Die asymptotischen Lösungen für sowohl starkes Ausblasen als auch Absaugen werden abgeleitet und eine gute Übereinstimmung mit den numerischen Lösungen festgestellt.

ВЛИЯНИЕ ВДУВА И ОТСОСА НА ПОГРАНИЧНЫЕ СЛОИ ПРИ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ

Аннотация — Рассматривается влияние вдува и отсоса на пограничные слои тел произвольной формы при естественной конвекции. Получено численное решение для горизонтального круглого цилиндра при постоянном вдуве и отсосе. Для некоторых форм тели скоростей вдува и отсоса получены решения подобия для уравнений пограничного слоя. Найдены асимптотические решения для интенсивного вдува и отсоса, хорошо согласующиеся с численными результатами.